

# ENE4014: Programming Languages

## Lecture 15 — Automatic Type Inference (3)

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## Finding a Solution of Type Equations

Find the values of type variables that make all the equations true.

$$\underbrace{\underbrace{\underbrace{\underbrace{\text{proc } (f)}_{t_f} \text{ proc } (x)}_{t_x} ((f \ 3) - (f \ x))}_{t_2}}_{t_1}}_{t_0}$$

Equations	Solution
$t_0 = t_f \rightarrow t_1$	$t_0 = (\text{int} \rightarrow \text{int}) \rightarrow (\text{int} \rightarrow \text{int})$
$t_1 = t_x \rightarrow t_2$	$t_1 = \text{int} \rightarrow \text{int}$
$t_3 = \text{int}$	$t_2 = \text{int}$
$t_4 = \text{int}$	$t_3 = \text{int}$
$t_2 = \text{int}$	$t_4 = \text{int}$
$t_f = \text{int} \rightarrow t_3$	$t_f = \text{int} \rightarrow \text{int}$
$t_x = t_x \rightarrow t_4$	$t_x = \text{int}$

Static type systems find such a solution using *unification algorithm*.

## Example 1

The calculation is split into equations to be solved and substitution found so far. Initially, the substitution is empty:

	Equations	Substitution
$t_0$	$= t_f \rightarrow t_1$	
$t_1$	$= t_x \rightarrow t_2$	
$t_3$	$= \text{int}$	
$t_4$	$= \text{int}$	
$t_2$	$= \text{int}$	
$t_f$	$= \text{int} \rightarrow t_3$	
$t_f$	$= t_x \rightarrow t_4$	

## Example 1

Consider each equation in turn. If the equation's left-hand side is a variable, move it to the substitution:

Equations	Substitution
$t_1 = t_x \rightarrow t_2$	$t_0 = t_f \rightarrow t_1$
$t_3 = \text{int}$	
$t_4 = \text{int}$	
$t_2 = \text{int}$	
$t_f = \text{int} \rightarrow t_3$	
$t_f = t_x \rightarrow t_4$	

## Example 1

Move the next equation to the substitution and propagate the information to the existing substitution (i.e., substitute the right-hand side for each occurrence of  $t_1$ ):

Equations	Substitution
$t_3 = \text{int}$	$t_0 = t_f \rightarrow (t_x \rightarrow t_2)$
$t_4 = \text{int}$	$t_1 = t_x \rightarrow t_2$
$t_2 = \text{int}$	
$t_f = \text{int} \rightarrow t_3$	
$t_f = t_x \rightarrow t_4$	

## Example 1

Same for the next three equations:

Equations	Substitution
$t_4 = \text{int}$	$t_0 = t_f \rightarrow (t_x \rightarrow t_2)$
$t_2 = \text{int}$	$t_1 = t_x \rightarrow t_2$
$t_f = \text{int} \rightarrow t_3$	$t_3 = \text{int}$
$t_f = t_x \rightarrow t_4$	

  

Equations	Substitution
$t_2 = \text{int}$	$t_0 = t_f \rightarrow (t_x \rightarrow t_2)$
$t_f = \text{int} \rightarrow t_3$	$t_1 = t_x \rightarrow t_2$
$t_f = t_x \rightarrow t_4$	$t_3 = \text{int}$
	$t_4 = \text{int}$

  

Equations	Substitution
$t_f = \text{int} \rightarrow t_3$	$t_0 = t_f \rightarrow (t_x \rightarrow \text{int})$
$t_f = t_x \rightarrow t_4$	$t_1 = t_x \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$

## Example 1

Consider the next equation  $t_f = \text{int} \rightarrow t_3$ . The equation contains  $t_3$ , which is already bound to  $\text{int}$  in the substitution. Substitute  $\text{int}$  for  $t_3$  in the equation. This is called *applying* the substitution to the equation.

Equations	Substitution
$t_f = \text{int} \rightarrow \text{int}$	$t_0 = t_f \rightarrow (t_x \rightarrow \text{int})$
$t_f = t_x \rightarrow t_4$	$t_1 = t_x \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$

Move the resulting equation to the substitution and update it.

Equations	Substitution
$t_f = t_x \rightarrow t_4$	$t_0 = (\text{int} \rightarrow \text{int}) \rightarrow (t_x \rightarrow \text{int})$
	$t_1 = t_x \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \rightarrow \text{int}$

## Example 1

Apply the substitution to the equation:

Equations	Substitution
$\text{int} \rightarrow \text{int} = t_x \rightarrow \text{int}$	$t_0 = (\text{int} \rightarrow \text{int}) \rightarrow (t_x \rightarrow \text{int})$
	$t_1 = t_x \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \rightarrow \text{int}$

If neither side of the equation is a variable, simplify the equation by yielding two new equations:

Equations	Substitution
$\text{int} = t_x$	$t_0 = (\text{int} \rightarrow \text{int}) \rightarrow (t_x \rightarrow \text{int})$
$\text{int} = \text{int}$	$t_1 = t_x \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \rightarrow \text{int}$

## Example 1

Switch the sides of the first equation and move it to the substitution:

Equations	Substitution
$\text{int} = \text{int}$	$t_0 = (\text{int} \rightarrow \text{int}) \rightarrow (\text{int} \rightarrow \text{int})$
	$t_1 = \text{int} \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \rightarrow \text{int}$
	$t_x = \text{int}$

The final substitution is the solution of the original equations.

## Example 2

$$\text{proc } \underbrace{(f)}_{t_f} \underbrace{(f \ 11)}_{t_1}$$

$t_0$

$$t_0 = t_f \rightarrow t_1$$

$$t_f = \text{int} \rightarrow t_1$$

## Example 2

1

Equations	Substitution
$t_0 = t_f \rightarrow t_1$	
$t_f = \text{int} \rightarrow t_1$	

2

Equations	Substitution
$t_f = \text{int} \rightarrow t_1$	$t_0 = t_f \rightarrow t_1$

3

Equations	Substitution
	$t_0 = (\text{int} \rightarrow t_1) \rightarrow t_1$
	$t_f = \text{int} \rightarrow t_1$

The type is *polymorphic* in  $t_1$ .

## Example 3

if  $\underbrace{x}_{t_x}$  then  $\underbrace{(x - 1)}_{t_1}$  else 0

$\underbrace{\hspace{15em}}_{t_0}$

$t_x = \text{bool}$

$t_1 = t_0$

$\text{int} = t_0$

$t_x = \text{int}$

$t_1 = \text{int}$

## Example 3

The equations have no solutions because, during the unification algorithm, we encounter the following contradictory state:

Equations	Substitution
$\text{bool} = \text{int}$	$t_x = \text{bool}$
$t_1 = \text{int}$	$t_1 = \text{int}$
	$t_0 = \text{int}$

Because `bool` and `int` cannot be equal, there is no solution to the equations.

## Example 4

$$\text{proc } \underbrace{(f)}_{t_f} \underbrace{(\text{iszero } \underbrace{(f f)}_{t_2})}_{t_1}$$
$$\underbrace{\hspace{15em}}_{t_0}$$
$$t_0 = t_f \rightarrow t_1$$
$$t_1 = \text{bool}$$
$$t_2 = \text{int}$$
$$t_f = t_f \rightarrow t_2$$

## Example 4

Solving as usual, we encounter a problem:

Equations	Substitution
$t_f = t_f \rightarrow \text{int}$	$t_0 = t_f \rightarrow \text{bool}$
	$t_1 = \text{bool}$
	$t_2 = \text{int}$

- There is no type  $t_f$  that satisfies the equation, because the right-hand side of the equation is always larger than the left.
- If we ever deduce an equation of the form  $t = \dots t \dots$  where the type variable  $t$  occurs in the right-hand side, we must conclude that there is no solution. This is called *occurrence check*.

# Unification Algorithm

For each equation in turn,

- Apply the current substitution to the equation.
- If the equation is always true (e.g.  $\text{int} = \text{int}$ ), discard it.
- If the left- and right-hand sides are contradictory (e.g.  $\text{bool} = \text{int}$ ), the algorithm fails.
- If neither side is a variable (e.g.  $\text{int} \rightarrow t_1 = t_2 \rightarrow \text{bool}$ ), simplify the equation, which eventually generates an equation whose left- or right-hand side is a variable.
- If the left-hand side is not a variable, switch the sides.
- If the left-hand side variable occurs in the right-hand side, the algorithm fails.
- Otherwise, move it to the substitution and substitute the right-hand side for each occurrence of the variable in the substitution.

# Exercise 1

let  $x = 4$  in  $(x \ 3)$

## Exercise 2

let  $f = \text{proc } (z) z \text{ in proc } (x) ((f x) - 1)$

## Exercise 3

let  $p = \text{iszero } 1$  in if  $p$  then 88 else 99

## Exercise 4

```
let f = proc (x) x in if (f (iszero0)) then (f 11) else (f 22)
```

## Substitution

Solutions of type equations are represented by substitution:

$$S \in \mathit{Subst} = \mathit{TyVar} \rightarrow \mathit{T}$$

Applying a substitution to a type:

$$\begin{aligned} S(\mathit{int}) &= \mathit{int} \\ S(\mathit{bool}) &= \mathit{bool} \\ S(\alpha) &= \begin{cases} t & \text{if } \alpha \mapsto t \in S \\ \alpha & \text{otherwise} \end{cases} \\ S(T_1 \rightarrow T_2) &= S(T_1) \rightarrow S(T_2) \end{aligned}$$

## Example

Applying the substitution

$$S = \{t_1 \mapsto \text{int}, t_2 \mapsto \text{int} \rightarrow \text{int}\}$$

to to the type  $(t_1 \rightarrow t_2) \rightarrow (t_3 \rightarrow \text{int})$ :

$$\begin{aligned} & S((t_1 \rightarrow t_2) \rightarrow (t_3 \rightarrow \text{int})) \\ &= S(t_1 \rightarrow t_2) \rightarrow S(t_3 \rightarrow \text{int}) \\ &= (S(t_1) \rightarrow S(t_2)) \rightarrow (S(t_3) \rightarrow S(\text{int})) \\ &= (\text{int} \rightarrow (\text{int} \rightarrow \text{int})) \rightarrow (t_3 \rightarrow \text{int}) \end{aligned}$$

# Unification

Update the current substitution with equality  $t_1 \doteq t_2$ .

**unify** :  $T \times T \times \text{Subst} \rightarrow \text{Subst}$

$$\mathbf{unify}(\text{int}, \text{int}, S) = S$$

$$\mathbf{unify}(\text{bool}, \text{bool}, S) = S$$

$$\mathbf{unify}(\alpha, \alpha, S) = S$$

$$\mathbf{unify}(\alpha, t, S) = \begin{cases} \text{fail} & \alpha \text{ occurs in } t \\ \text{extend } S \text{ with } \alpha \doteq t & \text{otherwise} \end{cases}$$

$$\mathbf{unify}(t, \alpha, S) = \mathbf{unify}(\alpha, t, S)$$

$$\mathbf{unify}(t_1 \rightarrow t_2, t'_1 \rightarrow t'_2, S) = \text{let } S' = \mathbf{unify}(t_1, t'_1, S) \text{ in} \\ \text{let } S'' = \mathbf{unify}(S'(t_2), S'(t'_2), S') \text{ in} \\ S''$$

$$\mathbf{unify}(-, -, -) = \text{fail}$$

## Exercises

- $\text{unify}(\alpha, \text{int} \rightarrow \text{int}, \emptyset) =$
- $\text{unify}(\alpha, \text{int} \rightarrow \alpha, \emptyset) =$
- $\text{unify}(\alpha \rightarrow \beta, \text{int} \rightarrow \text{int}, \emptyset) =$
- $\text{unify}(\alpha \rightarrow \beta, \text{int} \rightarrow \alpha, \emptyset) =$

# Solving Equations

**unifyall** :  $TyEqn \rightarrow Subst \rightarrow Subst$

**unifyall**( $\emptyset, S$ ) =  $S$

**unifyall**(( $t_1 \doteq t_2$ )  $\wedge$   $u, S$ ) = let  $S' = \mathbf{unify}(S(t_1), S(t_2), S)$   
in **unifyall**( $u, S'$ )

Let  $\mathcal{U}$  be the final unification algorithm:

$\mathcal{U}(u) = \mathbf{unifyall}(u, \emptyset)$

## **typeof** : $E \rightarrow T$

The final type inference algorithm that composes equation derivation ( $\mathcal{V}$ ) and equation solving ( $\mathcal{U}$ ):

$$\begin{aligned} \mathbf{typeof}(E) = & \\ & \mathbf{let } S = \mathcal{U}(\mathcal{V}(\emptyset, E, \alpha)) \quad (\text{new } \alpha) \\ & \mathbf{in } S(\alpha) \end{aligned}$$

## Examples

- `typeof((proc (x) x) 1)`

## Examples

- `typeof((proc (x) x) 1)`
- `typeof(let x = 1 in proc(y) (x + y))`

# Summary

Automatic type inference:

- derive type equations from the program text, and
- solve the equations by unification algorithm.