Homework 1
CSE6049 Program Analysis, Spring 2021

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due: 4/05(Mon), email-to-TA (bbumbuul@yahoo.com)

Exercise 1. Consider a set $T(\ni t)$ inductively defined as follows:

$$t \to \cdot \mid /t,t/ \mid /t,t,t/$$

Let $c(t)$ denote the number of occurrences of ",″ in t, and $s(t)$ denote the numbers of occurrences of "/″ in t.

Prove the following property over every $t \in T$:

$$s(t) \geq c(t)$$

□

Proof. Proof by structural induction.

Base case: $t = \cdot$:

$$\begin{array}{rcl} s(t) & = & 0 \\ & \geq & 0 \\ & = & c(t) \end{array}$$

Inductive case 1) $t = /t_1, t_2/$:
Inductive hypothesis: $s(t_1) \geq c(t_1)$ and $s(t_2) \geq c(t_2)$.

$$\begin{array}{rcll} s(t) & = & s(t_1) + s(t_2) + 2 & \\ & \geq & c(t_1) + c(t_2) + 2 & \text{(by inductive hypothesis)} \\ & \geq & c(t_1) + c(t_2) + 1 & \\ & = & c(t) & \end{array}$$

Inductive case 1) $t = /t_1, t_2, t_3/$:
Inductive hypothesis: $s(t_1) \geq c(t_1)$, $s(t_2) \geq c(t_2)$, and $s(t_3) \geq c(t_3)$.

$$\begin{array}{rcll} s(t) & = & s(t_1) + s(t_2) + s(t_3) + 2 & \\ & \geq & c(t_1) + c(t_2) + c(t_3) + 2 & \text{(by inductive hypothesis)} \\ & = & c(t) & \end{array}$$

□

Exercise 2. Consider the set of integer arithmetic expressions which is inductively defined as follows:

$$e \to x \mid e + e \mid e \times e \mid e \,?\, e\; e$$

where $e_1\ ?e_2\ e_3$ is a conditional expression which evaluates to e_3 (resp. e_2) if e_1 evaluates to zero (resp. non-zero).

Prove the following property over every arithmetic expresesion e: if every variable that appears in e holds a multiple of n, the evaluation result of e is also a multiple of n. For example, if $x = 4$ and $y = 2$ (both variables hold a multiple of 2), $x + y$ evaluates to 6 which is also a multiple of 2. □

Proof. Proof by structural induction. Let $[\![e]\!]$ denote the evaluation result of e.

Base case) $e = x$:

By the assumption that every variable in e holds a multiple of n, e holds a multiple of n.

Inductive case 1) $e = e_1 + e_2$:

Inductive hypothesis: $[\![e_1]\!] = nk_1$ and $[\![e_2]\!] = nk_2$ for some $k_1, k_2 \in \mathbb{Z}$.

$[\![e]\!] = [\![e_1]\!] + [\![e_2]\!] = n(k_1 + k_2)$. Therefore, e holds a multiple of n.

Inductive case 2) $e = e_1 \times e_2$:

Inductive hypothesis: $[\![e_1]\!] = nk_1$ and $[\![e_2]\!] = nk_2$ for some $k_1, k_2 \in \mathbb{Z}$.

$[\![e]\!] = [\![e_1]\!] \times [\![e_2]\!] = n \times n(k_1 \times k_2)$. Therefore, e holds a multiple of n.

Inductive case 3) $e = e_1?\ e_2\ e_3$:

Inductive hypothesis: $[\![e_1]\!] = nk_1$, $[\![e_2]\!] = nk_2$, and $[\![e_3]\!] = nk_3$ for some $k_1, k_2, k_3 \in \mathbb{Z}$.

$[\![e]\!] = [\![e_2]\!] = nk_2$ if $[\![e_1]\!] \neq 0$.

$[\![e]\!] = [\![e_3]\!] = nk_3$ if $[\![e_1]\!] = 0$.

Therefore, no matter which value e_1 evaluates to, e holds a multiple of n.

□

Exercise 3. Find the least fixpoint for each of the following functions.

- $\lambda x.\ 1 \quad \in \mathbb{Z} \to \mathbb{Z}$
- $\lambda x.\ x \quad \in \mathbb{Z} \to \mathbb{Z}$
- $\lambda x.\ x + 1 \quad \in \mathbb{Z} \cup \{\infty\} \to \mathbb{Z} \cup \{\infty\}$
- $\lambda f.\ (\lambda x.\ if\ x = 0\ then\ 0\ else\ x + f(x - 1)) \quad \in (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$
- $\lambda X.\ \{\epsilon\} \cup \{ax \mid x \in X\} \quad \in 2^S \to 2^S$ where S is the set of finite strings and 2^A denotes the powerset of A for set A.

□

Solutions:

- 1
- any integer

- ∞
- $\lambda x.\ \frac{x(x+1)}{2}$
- $\{a^i \mid i \geq 0\} = \{\epsilon, a, aa, aaa, \cdots\}$

Exercise 4. Prove the following:

Given two CPOs $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$, $(D, \sqsubseteq)$ is a CPO where

$$D = D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1, d_2 \in D_2\}$$

and

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \iff (d_1 \sqsubseteq_1 d_1') \wedge (d_2 \sqsubseteq_1 d_2').$$

□

Proof. Let say we have a chain in D which is $(x_0, y_0) \sqsubseteq (x_1, y_1) \sqsubseteq (x_2, y_2) \cdots$ where $\forall i.\ x_i \in D_1$ and $y_i \in D_2$. We will show that the least upper bound $\bigsqcup_{i \geq 0}(x_i, y_i)$ is in D.

We define $\bigsqcup_{i \geq 0}(x_i, y_i)$ to be $(\bigsqcup_{i \geq 0} x_i, \bigsqcup_{i \geq 0} y_i)$. Here, $(\bigsqcup_{i \geq 0} x_i, \bigsqcup_{i \geq 0} y_i) \in D$ because $\bigsqcup_{i \geq 0} x_i \in D_1$ and $\bigsqcup_{i \geq 0} y_i \in D_2$ as D_1 and D_2 are CPOs.

□