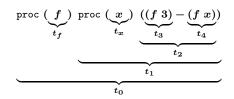ENE4014: Programming Languages

Lecture 15 — Automatic Type Inference (3)

Woosuk Lee
2024 Spring

Finding a Solution of Type Equations

Find the values of type variables that make all the equations true.

$$\text{proc } (\underbrace{f}_{t_f}) \text{ proc } (\underbrace{x}_{t_x}) \underbrace{((\underbrace{f\ 3}_{t_3}) - \underbrace{(f\ x)}_{t_4}))}_{t_2}$$

with t_1 and t_0 bracketing the larger expressions.

Equations			Solution		
t_0	$=$	$t_f \rightarrow t_1$	t_0	$=$	$(\text{int} \rightarrow \text{int}) \rightarrow (\text{int} \rightarrow \text{int})$
t_1	$=$	$t_x \rightarrow t_2$	t_1	$=$	$\text{int} \rightarrow \text{int}$
t_3	$=$	int	t_2	$=$	int
t_4	$=$	int	t_3	$=$	int
t_2	$=$	int	t_4	$=$	int
t_f	$=$	$\text{int} \rightarrow t_3$	t_f	$=$	$\text{int} \rightarrow \text{int}$
t_f	$=$	$t_x \rightarrow t_4$	t_x	$=$	int

Static type systems find such a solution using *unification algorithm*.

Example 1

The calculation is split into equations to be solved and substitution found
so far. Initially, the substitution is empty:

Equations			Substitution
t_0	$=$	$t_f \to t_1$	
t_1	$=$	$t_x \to t_2$	
t_3	$=$	int	
t_4	$=$	int	
t_2	$=$	int	
t_f	$=$	int $\to t_3$	
t_f	$=$	$t_x \to t_4$	

Example 1

Consider each equation in turn. If the equation's left-hand side is a variable, move it to the substitution:

Equations	Substitution
$t_1 = t_x \to t_2$	$t_0 = t_f \to t_1$
$t_3 = \text{int}$	
$t_4 = \text{int}$	
$t_2 = \text{int}$	
$t_f = \text{int} \to t_3$	
$t_f = t_x \to t_4$	

Example 1

Move the next equation to the substitution and propagate the information to the existing substitution (i.e., substitute the right-hand side for each occurrence of t_1):

Equations	Substitution
$t_3 = \text{int}$	$t_0 = t_f \to (t_x \to t_2)$
$t_4 = \text{int}$	$t_1 = t_x \to t_2$
$t_2 = \text{int}$	
$t_f = \text{int} \to t_3$	
$t_f = t_x \to t_4$	

Example 1

Same for the next three equations:

Equations	Substitution
$t_4 = \text{int}$	$t_0 = t_f \rightarrow (t_x \rightarrow t_2)$
$t_2 = \text{int}$	$t_1 = t_x \rightarrow t_2$
$t_f = \text{int} \rightarrow t_3$	$t_3 = \text{int}$
$t_f = t_x \rightarrow t_4$	

Equations	Substitution
$t_2 = \text{int}$	$t_0 = t_f \rightarrow (t_x \rightarrow t_2)$
$t_f = \text{int} \rightarrow t_3$	$t_1 = t_x \rightarrow t_2$
$t_f = t_x \rightarrow t_4$	$t_3 = \text{int}$
	$t_4 = \text{int}$

Equations	Substitution
$t_f = \text{int} \rightarrow t_3$	$t_0 = t_f \rightarrow (t_x \rightarrow \text{int})$
$t_f = t_x \rightarrow t_4$	$t_1 = t_x \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$

Example 1

Consider the next equation $t_f = \text{int} \rightarrow t_3$. The equation contains t_3, which is already bound to int in the substitution. Substitute int for t_3 in the equation. This is called *applying* the substitution to the equation.

Equations	Substitution
$t_f = \text{int} \rightarrow \text{int}$	$t_0 = t_f \rightarrow (t_x \rightarrow \text{int})$
$t_f = t_x \rightarrow t_4$	$t_1 = t_x \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$

Move the resulting equation to the substitution and update it.

Equations	Substitution
$t_f = t_x \rightarrow t_4$	$t_0 = (\text{int} \rightarrow \text{int}) \rightarrow (t_x \rightarrow \text{int})$
	$t_1 = t_x \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \rightarrow \text{int}$

Example 1

Apply the substitution to the equation:

Equations	Substitution
$\text{int} \to \text{int} = t_x \to \text{int}$	$t_0 = (\text{int} \to \text{int}) \to (t_x \to \text{int})$
	$t_1 = t_x \to \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \to \text{int}$

If neither side of the equation is a variable, simplify the equation by yielding two new equations:

Equations	Substitution
$\text{int} = t_x$	$t_0 = (\text{int} \to \text{int}) \to (t_x \to \text{int})$
$\text{int} = \text{int}$	$t_1 = t_x \to \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \to \text{int}$

Example 1

Switch the sides of the first equation and move it to the substitution:

Equations	Substitution
$\text{int} = \text{int}$	$t_0 = (\text{int} \rightarrow \text{int}) \rightarrow (\text{int} \rightarrow \text{int})$
	$t_1 = \text{int} \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \rightarrow \text{int}$
	$t_x = \text{int}$

The final substitution is the solution of the original equations.

Example 2

$$\text{proc}\ (\underbrace{f}_{t_f})\ (\underbrace{f\ 11}_{t_1})$$
$$\underbrace{\phantom{\text{proc}\ (f)\ (f\ 11)}}_{t_0}$$

$$
\begin{aligned}
t_0 &= t_f \to t_1 \\
t_f &= \text{int} \to t_1
\end{aligned}
$$

Example 2

1

Equations	Substitution
$t_0 = t_f \rightarrow t_1$	
$t_f = \text{int} \rightarrow t_1$	

2

Equations	Substitution
$t_f = \text{int} \rightarrow t_1$	$t_0 = t_f \rightarrow t_1$

3

Equations	Substitution
	$t_0 = (\text{int} \rightarrow t_1) \rightarrow t_1$
	$t_f = \text{int} \rightarrow t_1$

The type is *polymorphic* in t_1.

Example 3

$$\underbrace{\text{if }\underbrace{x}_{t_x}\text{ then }\underbrace{(x-1)}_{t_1}\text{ else } 0}_{t_0}$$

$$
\begin{aligned}
t_x &= \text{bool} \\
t_1 &= t_0 \\
\text{int} &= t_0 \\
t_x &= \text{int} \\
t_1 &= \text{int}
\end{aligned}
$$

Example 3
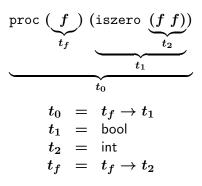
The equations have no solutions because, during the unification algorithm, we encounter the following contradictory state:

Equations			Substitution		
bool	=	int	t_x	=	bool
t_1	=	int	t_1	=	int
			t_0	=	int

Because bool and int cannot be equal, there is no solution to the equations.

Example 4

$$\text{proc } (\underbrace{f}_{t_f}) \ (\text{iszero } \underbrace{(f \ f)}_{t_2})$$

$$\underbrace{\hspace{3cm}}_{t_1}$$

$$\underbrace{\hspace{5cm}}_{t_0}$$

$$
\begin{aligned}
t_0 &= t_f \rightarrow t_1 \\
t_1 &= \text{bool} \\
t_2 &= \text{int} \\
t_f &= t_f \rightarrow t_2
\end{aligned}
$$

Example 4

Solving as usual, we encounter a problem:

Equations	Substitution
$t_f = t_f \rightarrow \text{int}$	$t_0 = t_f \rightarrow \text{bool}$
	$t_1 = \text{bool}$
	$t_2 = \text{int}$

- There is no type t_f that satisfies the equation, because the right-hand side of the equation is always larger than the left.

- If we ever deduce an equation of the form $t = \ldots t \ldots$ where the type variable t occurs in the right-hand side, we must conclude that there is no solution. This is called *occurrence check*.

Unification Algorithm

For each equation in turn,

- Apply the current substitution to the equation.
- If the equation is always true (e.g. int $=$ int), discard it.
- If the left- and right-hand sides are contradictory (e.g. bool $=$ int), the algorithm fails.
- If neither side is a variable (e.g. int $\rightarrow t_1 = t_2 \rightarrow$ bool), simplify the equation, which eventually generates an equation whose left- or right-hand side is a variable.
- If the left-hand side is not a variable, switch the sides.
- If the left-hand side variable occurs in the right-hand side, the algorithm fails.
- Otherwise, move it to the substitution and substitute the right-hand side for each occurrence of the variable in the substitution.

Exercise 1

$$\texttt{let } x = 4 \texttt{ in } (x\ 3)$$

Exercise 2

$$\texttt{let } f = \texttt{proc } (z) \; z \texttt{ in proc } (x) \; ((f \; x) - 1)$$

Exercise 3

let p = iszero $\mathbf{1}$ in if p then $\mathbf{88}$ else $\mathbf{99}$

Exercise 4

let f = proc (x) x in if $(f$ (iszero0)) then $(f$ 11) else $(f$ 22)

Substitution

Solutions of type equations are represented by substitution:

$$S \in Subst = TyVar \to T$$

Applying a substitution to a type:

$$
\begin{aligned}
S(\text{int}) &= \text{int} \\
S(\text{bool}) &= \text{bool} \\
S(\alpha) &= \begin{cases} t & \text{if } \alpha \mapsto t \in S \\ \alpha & \text{otherwise} \end{cases} \\
S(T_1 \to T_2) &= S(T_1) \to S(T_2)
\end{aligned}
$$

Example

Applying the substitution

$$S = \{t_1 \mapsto \text{int}, t_2 \mapsto \text{int} \rightarrow \text{int}\}$$

to to the type $(t_1 \rightarrow t_2) \rightarrow (t_3 \rightarrow \text{int})$:

$$S((t_1 \rightarrow t_2) \rightarrow (t_3 \rightarrow \text{int}))$$
$$= S(t_1 \rightarrow t_2) \rightarrow S(t_3 \rightarrow \text{int})$$
$$= (S(t_1) \rightarrow S(t_2)) \rightarrow (S(t_3) \rightarrow S(\text{int}))$$
$$= (\text{int} \rightarrow (\text{int} \rightarrow \text{int})) \rightarrow (t_3 \rightarrow \text{int})$$

Unification

Update the current substitution with equality $t_1 \doteq t_2$.

$$\text{unify} : T \times T \times Subst \to Subst$$

$$
\begin{aligned}
\text{unify}(\text{int}, \text{int}, S) &= S \\
\text{unify}(\text{bool}, \text{bool}, S) &= S \\
\text{unify}(\alpha, \alpha, S) &= S \\
\text{unify}(\alpha, t, S) &= \begin{cases} \text{fail} & \alpha \text{ occurs in } t \\ \text{extend } S \text{ with } \alpha \doteq t & \text{otherwise} \end{cases} \\
\text{unify}(t, \alpha, S) &= \text{unify}(\alpha, t, S) \\
\text{unify}(t_1 \to t_2, t_1' \to t_2', S) &= \text{let } S' = \text{unify}(t_1, t_1', S) \text{ in} \\
&\quad \text{let } S'' = \text{unify}(S'(t_2), S'(t_2'), S') \text{ in} \\
&\quad\quad S'' \\
\text{unify}(\_, \_, \_) &= \text{fail}
\end{aligned}
$$

Exercises

- **unify**$(\alpha, \text{int} \rightarrow \text{int}, \emptyset) =$
- **unify**$(\alpha, \text{int} \rightarrow \alpha, \emptyset) =$
- **unify**$(\alpha \rightarrow \beta, \text{int} \rightarrow \text{int}, \emptyset) =$
- **unify**$(\alpha \rightarrow \beta, \text{int} \rightarrow \alpha, \emptyset) =$

Solving Equations

$$\textbf{unifyall} : TyEqn \rightarrow Subst \rightarrow Subst$$

$$\textbf{unifyall}(\emptyset, S) = S$$

$$\textbf{unifyall}((t_1 \doteq t_2) \wedge u, S) = \begin{aligned} &\text{let } S' = \textbf{unify}(S(t_1), S(t_2), S) \\ &\text{in } \textbf{unifyall}(u, S') \end{aligned}$$

Let $\mathcal{U}$ be the final unification algorithm:

$$\mathcal{U}(u) = \textbf{unifyall}(u, \emptyset)$$

typeof : $E \to T$

The final type inference algorithm that composes equation derivation ($\mathcal{V}$) and equation solving ($\mathcal{U}$):

$$\begin{aligned}
\textbf{typeof}(E) = \\
\quad \textbf{let } S = \mathcal{U}(\mathcal{V}(\emptyset, E, \alpha)) \quad \text{(new } \alpha) \\
\quad \textbf{in } S(\alpha)
\end{aligned}$$

Examples

- **typeof**$((\text{proc } (x)\ x)\ 1)$

Examples

- **typeof**((proc (x) x) 1)
- **typeof**(let $x = 1$ in proc(y) $(x + y)$)

Summary

Automatic type inference:

- derive type equations from the program text, and
- solve the equations by unification algorithm.