Homework 1
ENE4014 Programming Languages, Spring 2024
due: 4/01 (Mon)

- Submit your answer (as a hard copy) to the instructor after the class on 4/1 (do not forget to write your name and student ID!).

Exercise 1. (3 points) Write inductive definitions of the following set

$$S = \{3n + 2 \mid n \in \mathbb{N}\}$$

in all three styles (top-down, bottom-up, and rules of inference).

- Top-down:
  - $2 \in S$
  - $n \in S$ if $n - 3 \in S$
- Bottom-up:
  - $2 \in S$
  - if $n \in S$, $n + 3 \in S$
- Rules of inference: $\dfrac{}{2(\in S)} \qquad \dfrac{n(\in S)}{n + 3(\in S)}$

Exercise 2. (3 points) Write inductive definitions of the following set

$$S = \{2n + 3m + 1 \mid n, m \in \mathbb{N}\}$$

in all three styles (top-down, bottom-up, and rules of inference).

- Top-down:
  - $1 \in S$
  - $n \in S$ if $n-2 \in S$ or $n-3 \in S$
- Bottom-up:
  - $1 \in S$
  - if $n \in S$, then $n+2 \in S$ and $n+3 \in S$
- Rules of inference: $\dfrac{}{1(\in S)} \qquad \dfrac{n(\in S)}{n+2(\in S)} \qquad \dfrac{n(\in S)}{n+3(\in S)}$

Exercise 3. (4 points) What sets of strings are defined by the following rules?

1. $\dfrac{}{\mathtt{a}} \qquad \dfrac{}{\mathtt{b}} \qquad \dfrac{x_1 \quad x_2}{x_1 \mathtt{c} x_2}$
2. $\dfrac{}{(\,)} \qquad \dfrac{x}{((x))}$

- $S = \{\mathtt{a}, \mathtt{b}, \mathtt{aca}, \mathtt{acb}, \mathtt{bca}, \mathtt{bcb}, \mathtt{acaca}, \cdots\}$
- $S = \{(), ((())), ((((())))) \cdots\}$ or $S = \{(^i)^i \mid i = 2k+1, k \in \mathbb{N}\}$

Exercise 4. (2 points) Consider a set of binary trees inductively defined as follows:

$$\begin{array}{rcl} t & \rightarrow & n \quad (n \in \mathbb{N}) \\ & \mid & (n, t, t) \end{array}$$

Write a derivation tree for the binary tree $(5, 4, (1, 2, 3))$.

$$\dfrac{\dfrac{}{4} \quad \dfrac{\dfrac{}{2} \quad \dfrac{}{3}}{(1,2,3)}}{(5,4,(1,2,3))}$$

Exercise 5. (4 points) Consider a set of lists inductively defined as follows:

$$\begin{array}{lcl} t & \rightarrow & \texttt{nil} \\ & | & n \cdot m \cdot t \quad (n, m \in \mathbb{N}) \end{array}$$

Prove every list contains an even number of integers.

More formally, let $i(t)$ denote the number of integers that appear in list t. Our goal is to prove the following property over every $t \in T$:

$$i(t) = 2k \text{ for some } k \in \mathbb{N}$$

Proof. Proof by structural induction.

Base case: $t = \texttt{nil}$: $i(t) = 0$ which proves the theorem as $k = 0$.

Inductive case) $t = n \cdot m \cdot t'$:

Inductive hypothesis: $i(t') = 2k'$ for some $k' \in \mathbb{N}$.

$$\begin{array}{rcl} i(t) & = & 2 + i(t') \\ & = & 2 + 2k' \\ & = & 2(k' + 1) \end{array}$$

□

Exercise 6. (4 points) Consider the set of binary trees in Exercise 4. Prove every tree contains an odd number of integers.

More formally, let $i(t)$ denote the number of integers that appear in tree t. Our goal is to prove the following property over every $t \in T$:

$$i(t) = 2k + 1 \text{ for some } k \in \mathbb{N}.$$

Proof. Proof by structural induction.

Base case: $t = n$: $i(t) = 1$ which proves the theorem as $k = 0$.

Inductive case: $t = (n, t_1, t_2)$:

Inductive hypothesis: $i(t_1) = 2k_1 + 1, i(t_2) = 2k_2 + 1$ for some $k_1, k_2 \in \mathbb{N}$.

$$\begin{array}{rcl} i(t) & = & 1 + i(t_1) + i(t_2) \\ & = & 1 + 2k_1 + 1 + 2k_2 + 1 \\ & = & 2(k_1 + k_2 + 1) + 1 \end{array}$$

which proves the theorem for the inductive case.

□